AD-753 459

STABILITY CRITERIA FOR ELASTIC MATERIALS

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Lehigh University

Prepared for:

Office of Naval Research

December 1972

CENTER FOR THE APPLICATION OF MATHEMATICS

LEHIGH UNIVERSITY

BETHLEHEM, PA.

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by

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TECHNICAL REPORT NO. CAM-100-16 December, 1972

OFFICE OF NAVAL RESEARCH CONTRACT NO. N00014-67-A-0370-0001

22

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Center for the Application of Mathematics Lehigh University	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP

3. REPORT TITLE

Stability criteria for elastic materials

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Unclassified Technical Report December, 1972

5. AUTHOR(S) (First name, middle initial, last name)

K.N. Sawyers and R.S. Rivlin

6. REPORT DATE	7a. TOTAL NO OF PAGES	7b. NO. OF REFS
December 1972	22	3

8a. CONTRACT OR GRANT NO. N00014-67-A-0370-0001	9a. ORIGINATOR'S REPORT NUMBER(S) CAM-100-16
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

13. ABSTRACT

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Stability, elastic materials, incompressible materials, finite deformations						

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Stability Criteria for Elastic Materials

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Abstract

Conditions are derived which are necessary for stability of incompressible elastic materials. These are obtained by considering small shearing displacements superimposed on a finitely deformed state of the material.

1. Introduction.

In the classical elasticity theory for infinitesimal deformations, certain restrictions must be placed on the elastic moduli to ensure that the material be stable. For example, in the case when the material is isotropic and incompressible, the shear modulus μ must be positive. This condition ensures that the velocity of propagation of a plane sinusoidal shear wave be real. If it were violated, a plane spatially-sinusoidal, shear disturbance, of initially infinitesimal amplitude, would increase in amplitude with time even though no forces are applied to the material. Accordingly, due to the presence of Brownian fluctuations, the material could not exist in the undeformed state.

Similarly, in the case of incompressible elastic materials which undergo finite deformations, material stability requires that the strain-energy function be such that, for each state of pure homogeneous deformation for which it is assumed to be valid, the velocities of plane sinusoidal shear waves of arbitrary direction of propagation and infinitesimal amplitude must be real.

It has been shown that this implies that the strain-energy function W per unit volume must satisfy the conditions

$$W_1 + \lambda_A^2 W_2 > 0$$

and

$$\frac{W_{11} + 2\lambda_A^2 W_{12} + \lambda_A^4 W_{22}}{W_1 + \lambda_A^2 W_2} > -\frac{1}{2}\,\frac{1}{I_1 - \lambda_A^2 - 2\lambda_A^{-1}}$$

where λ_A (A=1,2,3) are the principal extension ratios for the pure homogeneous deformation, W_A and W_{AB} (A,B = 1,2) denote the derivatives $\partial W/\partial I_A$ and $\partial^2 W/\partial I_A \partial I_B$ respectively of W with respect to the strain invariants I_1 and I_2 defined by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad , \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2$$

The first of these conditions, which was obtained by Ericksen [1], ensures that a shear wave, polarized in the principal direction corresponding to the extension ratio λ_A and propagated in an arbitrary direction perpendicular to this principal direction, shall have a real velocity of propagation. The second, which was derived by Sawyers and Rivlin [2], ensures that a shear wave, polarized in the plane perpendicular to the principal direction corresponding to the extension ratio λ_A and propagating in an arbitrary direction in this plane, shall have a real velocity of propagation.

In the present paper, it is shown that both of these conditions can be derived from a single assumption. We consider that an infinitesimal simple shear is superposed on the pure homogeneous deformation. The plane of shear is a principal plane for the pure homogeneous deformation and the direction of shear is an arbitrary direction in this plane. The assumption is made that the incremental shear modulus is positive.

2. Basic Equations

We consider deformations which result from the superposition of an infinitesimal deformation on a finite pure homogeneous deformation. Let ξ_i denote the coordinates in a rectangular cartesian system x of a generic particle of the body in its undeformed state. Let x_i be the coordinates of the same particle in the deformed state. We may then write

$$x_i = X_i + \varepsilon u_i \tag{1}$$

where ε is a small parameter whose squares and higher powers may be neglected, and

$$X_1 = \lambda_1 \xi_1 \quad , \quad X_2 = \lambda_2 \xi_2 \quad , \quad X_3 = \lambda_3 \xi_3 \tag{2}$$

The λ's in Eqn. 2 are the extension ratios associated with the finite deformation. In Eqn. 1, the displacements u_i are, in general, regarded as functions of X_1, X_2, X_3 .

The finger strain tensor $\bar{C}_{ij}$, associated with the resultant deformation, is given by*

$$\bar{C}_{ij} = \frac{\partial x_i}{\partial \xi_m} \frac{\partial x_j}{\partial \xi_m} = \frac{\partial X_k}{\partial \xi_m} \frac{\partial X_\ell}{\partial \xi_m} x_{i,k} x_{j,\ell} \tag{3}$$

---

* The usual summation convention applies to repeated subscripts. The comma notation $,i$ is used to denote partial differentiation with respect to X_i .

We write

$$\overline{C}_{ij} = C_{ij} + \varepsilon c_{ij} \tag{4}$$

and substitute from Eqns. 1 and 2 into 3 to find

$$C_{11} = \lambda_1^2 , \quad C_{22} = \lambda_2^2 , \quad C_{33} = \lambda_3^2 ,$$

$$C_{ij} = 0 \quad \text{for } i \neq j \tag{5}$$

and

$$c_{ij} = \lambda_1^2(\delta_{i1}u_{j,1} + \delta_{j1}u_{i,1}) + \lambda_2^2(\delta_{i2}u_{j,2} + \delta_{j2}u_{i,2})$$

$$+ \lambda_3^2(\delta_{i3}u_{j,3} + \delta_{j3}u_{i,3}) \tag{6}$$

where δ_{ij} is the Kronecker delta

Three scalar invariants $\overline{I}_A$ (A=1,2,3) of the tensor $\overline{C}_{ij}$ are given by

$$\overline{I}_1 = \overline{C}_{ii} , \quad \overline{I}_2 = \frac{1}{2}(\overline{C}_{ii}\overline{C}_{jj} - \overline{C}_{ij}\overline{C}_{ij}) , \quad \overline{I}_3 = \det (\overline{C}_{ij}) \tag{7}$$

Writing

$$\overline{I}_A = I_A + \varepsilon i_A , \quad A = 1,2,3 \tag{8}$$

and substituting from Eqns. 4, 5 and 6 into 7, we find

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2 \;, \quad I_2 = \lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2+\lambda_1^2\lambda_2^2 \;, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{9}$$

and

$$\begin{aligned} i_1 &= 2\left\{\lambda_1^2 u_{1,1} + \lambda_2^2 u_{2,2} + \lambda_3^2 u_{3,3}\right\} \\ i_2 &= 2\left\{\lambda_1^2(\lambda_2^2+\lambda_3^2)u_{1,1} + \lambda_2^2(\lambda_3^2+\lambda_1^2)u_{2,2} + \lambda_3^2(\lambda_1^2+\lambda_2^2)u_{3,3}\right\} \\ i_3 &= 2\lambda_1^2\lambda_2^2\lambda_3^2(u_{1,1}+ u_{2,2}+ u_{3,3}) \end{aligned} \tag{10}$$

From Eqns. 8, 9 and 10, it follows that the condition $\bar{I}_3 = 1$ for a deformation to be isochoric yields

$$\lambda_1^2\lambda_2^2\lambda_3^2 = 1 \quad \text{and} \quad u_{k,k} = 0 \tag{11}$$

Let N_i and $\bar{N}_i$, respectively, denote the components of the unit vectors normal to a surface in the material before and after it has undergone the infinitesimal superposed deformation. Then (see [3], for example)

$$\bar{N}_i = N_i + \varepsilon n_i \tag{12}$$

where

$$n_i = N_i N_k N_\ell u_{k,\ell} - N_k u_{k,i} \tag{13}$$

From Eqn. 1 we readily obtain, neglecting terms of higher degree in ε than the first, the relation

$$\frac{\partial}{\partial x_j} = \frac{\partial X_k}{\partial x_j}\frac{\partial}{\partial X_k} = \frac{\partial}{\partial X_j} - \varepsilon u_{k,j}\frac{\partial}{\partial X_k} \tag{14}$$

The strain energy W per unit volume for an isotropic incompressible elastic material depends on the strain $\overline{C}_{ij}$ only through the invariants $\overline{I}_1$ and $\overline{I}_2$, so that $W = W(\overline{I}_1, \overline{I}_2)$. The Cauchy stress tensor $\overline{\Sigma}_{ij}$ is then given by

$$\overline{\Sigma}_{ij} = 2[(\overline{W}_1 + \overline{I}_1\overline{W}_2)\overline{C}_{ij} - \overline{W}_2\overline{C}_{ik}\overline{C}_{kj}] - \overline{P}\delta_{ij} \tag{15}$$

where $\overline{P}$ is an arbitrary hydrostatic pressure and we use the notation

$$\overline{W}_A = \partial W/\partial \overline{I}_A \quad , \quad A = 1,2 \tag{16}$$

Noting that the invariants given by Eqn. 9 are those associated with the finite deformation, we make use of Eqn. 8 and expand $\overline{W}_1$ and $\overline{W}_2$ in a power series about I_1 and I_2. Thus,

$$\begin{aligned} \overline{W}_1 &= W_1 + \varepsilon(W_{11}i_1 + W_{12}i_2) \\ \overline{W}_2 &= W_2 + \varepsilon(W_{12}i_1 + W_{22}i_2) \end{aligned} \tag{17}$$

where

$$W_A = \overline{W}_A(I_1, I_2) \ , \ W_{AB} = \left.\left(\frac{\partial^2 W}{\partial \overline{I}_A \partial \overline{I}_B}\right)\right|_{I_1, I_2} \ , \ A,B = 1,2 \tag{18}$$

Writing

$$\bar{\Sigma}_{ij} = \Sigma_{ij} + \varepsilon\sigma_{ij} \quad , \quad \bar{P} = P + \varepsilon p \tag{19}$$

and substituting from Eqns. 4, 8, 17 and 19 into 15 , we find

$$\Sigma_{ij} = 2[(W_1 + I_1W_2)C_{ij} - W_2C_{ik}C_{kj}] - P\delta_{ij} \tag{20}$$

and

$$\begin{aligned}\sigma_{ij} = 2[(W_1 + I_1W_2)c_{ij} - W_2(C_{ik}c_{kj} + c_{ik}C_{kj}) \\ + \{(W_2 + W_{11})i_1 + W_{12}(I_1i_1 + i_2) + W_{22}I_1i_2\}C_{ij} \\ - (W_{12}i_1 + W_{22}i_2)C_{ik}C_{kj}] - p\delta_{ij}\end{aligned} \tag{21}$$

The traction $\bar{T}_i$ acting on a surface whose unit normal is $\bar{N}_i$, and measured per unit area of that surface, is given by

$$\bar{T}_i = \bar{\Sigma}_{ij}\bar{N}_j \tag{22}$$

Writing

$$\bar{T}_i = T_i + \varepsilon t_i \tag{23}$$

and substituting from Eqns. 12 and 19 into 22 yields

$$T_i = \Sigma_{ij}N_j \quad , \quad t_i = \sigma_{ij}N_j + \Sigma_{ij}n_j \tag{24}$$

In the absence of body forces the equation of equilibrium is

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = 0 \tag{25}$$

Substitution from Eqns. 14 and 19 into this yields the two equations

$$\Sigma_{ij,j} = 0 \quad \text{and} \quad \sigma_{ij,j} - u_{k,j}\Sigma_{ij,k} = 0 \tag{26}$$

3. Some Preliminary Results

We now assume that the superposed infinitesimal deformation is a simple shear in the plane perpendicular to the 3-direction.

The displacement u_i associated with this deformation is given by

$$
\begin{aligned}
u_1 &= \kappa(X_2\cos\theta - X_1\sin\theta)\cos\theta \\
u_2 &= \kappa(X_2\cos\theta - X_1\sin\theta)\sin\theta \qquad (27) \\
u_3 &= 0
\end{aligned}
$$

where κ is the amount of shear and θ is the angle between the direction of shear and the x_1-axis, as shown in Fig.1. We remark that the displacement described in Eqn. 27 satisfies the incompressibility requirement given by Eqn.11.

If κ is positive, then the direction of shear is in the direction of the unit vector $\underset{\sim}{K}$ with components

$$K_1 = \cos\theta \ , \quad K_2 = \sin\theta \ , \quad K_3 = 0 \qquad (28)$$

Let $\underset{\sim}{N}$ be a unit vector in the plane of shear perpendicular to $\underset{\sim}{K}$, the components of $\underset{\sim}{N}$ being given by

$$N_1 = -\sin\theta \ , \ N_2 = \cos\theta \ , \ N_3 = 0 \qquad (29)$$

We consider a material plane in the body which has undergone the pure homogeneous deformation described by Eqn. 2. The unit normal $\bar{\underset{\sim}{N}}$ to this plane after the superposition of the infinitesimal shear is given by Eqn. 12, where, from Eqns. 13, 27 and 29,

$$\underset{\sim}{n} = 0 \tag{30}$$

We substitute from Eqns. 27 into 6 and 10 and obtain

$$\begin{aligned} c_{11} &= -2\lambda_1^2\kappa \sin\theta\cos\theta \ , \quad c_{22} = 2\lambda_2^2\kappa \sin\theta\cos\theta \ , \\ c_{12} &= \kappa(\lambda_2^2\cos^2\theta - \lambda_1^2\sin^2\theta) \ , \quad c_{23} = c_{31} = c_{33} = 0 \ , \\ i_2 &= \lambda_3^2 \ i_1 = 2\lambda_3^2\kappa \sin\theta\cos\theta(\lambda_2^2-\lambda_1^2) \end{aligned} \tag{31}$$

We substitute from Eqns. 5 and 9 in Eqn. 20 and obtain the components of the Cauchy stress associated with the finite homogeneous deformation. Thus

$$\begin{aligned} \Sigma_{11} &= 2\lambda_1^2 \left\{W_1 + (\lambda_2^2 + \lambda^2)W_2\right\} - P \\ \Sigma_{22} &= 2\lambda_2^2 \left\{W_1 + (\lambda_3^2 + \lambda_1^2)W_2\right\} - P \\ \Sigma_{33} &= 2\lambda_3^2 \left\{W_1 + (\lambda_1^2 + \lambda_2^2)W_2\right\} - P \\ \Sigma_{23} &= \Sigma_{31} = \Sigma_{12} = 0 \end{aligned} \tag{32}$$

Similarly, substitution from Eqns. 5, 9 and 31 in Eqn. 21 gives the additional stresses that arise from the deformation described by Eqn. 27. Thus,

$$\sigma_{11} = -4\kappa\lambda_1^2 \sin\theta\cos\theta \left[W_1 + \lambda_3^2 W_2 + (\lambda_1^2-\lambda_2^2)\left\{W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22} + \lambda_2^2(W_{12}+\lambda_3^2 W_{22})\right\}\right] - p$$

$$\sigma_{22} = 4\kappa\lambda_2^2 \sin\theta\cos\theta \left[W_1 + \lambda_3^2 W_2 - (\lambda_1^2-\lambda_2^2)\left\{W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22} + \lambda_1^2(W_{12}+\lambda_3^2 W_{22})\right\}\right] - p \quad (33)$$

$$\sigma_{33} = 4\kappa\lambda_3^2 \sin\theta\cos\theta(\lambda_2^2-\lambda_1^2)\left[W_2 + W_{11} + (\lambda_1^2+\lambda_2^2+\lambda_3^2)W_{12} + \lambda_3^2 W_{22}(\lambda_1^2+\lambda_2^2)\right] - p$$

$$\sigma_{12} = 2\kappa(\lambda_2^2\cos^2\theta - \lambda_1^2\sin^2\theta)\left[W_1 + \lambda_3^2 W_2\right]$$

$$\sigma_{23} = \sigma_{31} = 0$$

Bearing in mind that $\lambda_1, \lambda_2, \lambda_3$, κ and θ are constants, it follows that the stresses given by Eqns. 32 and 33 satisfy the equilibrium conditions of Eqn. 26 provided P and p are constants.

The traction acting on a plane with unit normal $\underset{\sim}{N}$ can be obtained by substituting from Eqn. 29 into 24. By making use of Eqn. 30 and the fact that certain components of the stresses vanish (see Eqns. 32 and 33), we obtain

$$T_1 = -\sin\theta\ \Sigma_{11}, \quad T_2 = \cos\theta\ \Sigma_{22}, \quad T_3 = 0 \quad (34)$$

and

$$t_1 = -\sin\theta\sigma_{11} + \cos\theta\sigma_{12}$$

$$t_2 = -\sin\theta\sigma_{12} + \cos\theta\sigma_{22} \tag{35}$$

$$t_3 = 0$$

Let $\overline{S}$ denote the component of the traction vector in the shear direction $\underset{\sim}{K}$. Then, from Eqn. 23,

$$\overline{S} = \underset{\sim}{\overline{T}} \cdot \underset{\sim}{K} = (T_j + \varepsilon t_j) K_j \tag{36}$$

Writing

$$\overline{S} = S + \varepsilon s \tag{37}$$

and substituting from Eqns. 28, 34, and 35 into 36, we obtain

$$\begin{aligned} S &= \sin\theta\cos\theta(\Sigma_{22} - \Sigma_{11}) \\ s &= \sin\theta\cos\theta(\sigma_{22} - \sigma_{11}) + \sigma_{12}(\cos^2\theta - \sin^2\theta) \end{aligned} \tag{38}$$

Expressions for S and s in terms of material properties can be obtained by substituting from Eqns. 32 and 33 into 38. Introducing the notation

$$\begin{aligned} M &= 2(W_1 + \lambda_3^2 W_2) \\ m &= 4(\lambda_1 + \lambda_2)^2 (W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}) \end{aligned} \tag{39}$$

we then obtain

$$S = (\lambda_2^2 - \lambda_1^2) M \sin\theta\cos\theta$$

$$s = \kappa\{M(\lambda_1^2\sin^2\theta + \lambda_2^2\cos^2\theta)$$

$$+ (\lambda_1 - \lambda_2)^2 m \sin^2\theta\cos^2\theta\} \quad (40)$$

4. Criteria for Material Stability

In order that the material be stable for given values of λ_1, λ_2, and λ_3, it is necessary that the superimposed ~action have a positive component in the direction of the superimposed shear deformation, i.e., the shear modulus must be positive.

According to Eqns. 36 and 37, s is the extra shear traction that is required to produce the deformation given by Eqn. 27. Since κ is the amount of shear, we define the shear modulus μ by

$$\mu = \tilde{}/\kappa \tag{41}$$

Then from Eqn. 40 we obtain

$$\mu = M(\lambda_1^2 \sin^2\theta + \lambda_2^2 \cos^2\theta) + (\lambda_1 - \lambda_2)^2 m \sin^2\theta \cos^2\theta \tag{42}$$

and it follows that a necessary condition for material stability is $\mu > 0$ for all possible values of θ. In particular, if θ assumes any of the values $(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$ then $\mu > 0$ if and only if

$$M > 0 \tag{43}$$

For the following discussion we assume that Eqn. 43 is satisfied.

Then if $\lambda_1 = \lambda_2$, it follows that $\mu > 0$ regardless of the value of m . But if $\lambda_1 \neq \lambda_2$ and if

$$m(\lambda_1-\lambda_2)^2 = - M(\lambda_1+\lambda_2)^2 \tag{44}$$

then Eqn. 42 becomes

$$\mu = M(\lambda_1\sin^2\theta-\lambda_2\cos^2\theta)^2 \tag{45}$$

and it follows that μ is non-negative for all values of θ and vanishes if

$$\tan\theta = \pm \sqrt{\lambda_2/\lambda_1} \tag{46}$$

It is clear from Eqn. 42 that for any fixed value of θ , (other than those for which either $\sin\theta$ or $\cos\theta$ vanish) μ increases as $m(\lambda_1-\lambda_2)^2$ increases and decreases as $m(\lambda_1-\lambda_2)^2$ decreases. Thus, the condition $\mu > 0$ for all values of θ is satisfied if

$$m > - M \frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2} \tag{47}$$

However, if $m(\lambda_1-\lambda_2)^2 < - M(\lambda_1+\lambda_2)^2$, then $\mu < 0$ for θ given by Eqn. 46, and we conclude that the material is inherently unstable.

We have derived two conditions that are necessary for material stability. By substitution from Eqn. 39 in 43 and 47, these are found to be

$$
\begin{aligned}
&W_1 + \lambda_3^2 W_2 > 0 \\
&W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22} > -\frac{1}{2}\frac{W_1 + \lambda_3^2 W_2}{(\lambda_1 - \lambda_2)^2}
\end{aligned}
\tag{48}
$$

Further conditions analogous to those in Eqn. 48 can be derived by considering superposed shearing deformations similar to that of Eqn. 27, for which the shear planes are the 23 and 31 planes, respectively. An analysis which parallels that given above then yields necessary conditions for material stability in the forms

$$
\begin{aligned}
&W_1 + \lambda_1^2 W_2 > 0 \\
&W_{11} + 2\lambda_1^2 W_{12} + \lambda_1^4 W_{22} > -\frac{1}{2}\frac{W_1 + \lambda_1^2 W_2}{(\lambda_2 - \lambda_3)^2}
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
&W_1 + \lambda_2^2 W_2 > 0 \\
&W_{11} + 2\lambda_2^2 W_{12} + \lambda_2^4 W_{22} > -\frac{1}{2}\frac{W_1 + \lambda_2^2 W_2}{(\lambda_3 - \lambda_1)^2}
\end{aligned}
\tag{50}
$$

The conditions given in Eqns. 48, 49 and 50 may be recast in the succinct forms shown in the Introduction by employing Eqns. 9 and 11.

Acknowledgement

This work was carried out with the support of the Office of Naval Research under Contract No. N00014-67-0370-0001 with Lehigh University.

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Figure Caption

Figure 1. Geometry of a small superimposed shearing deformation.

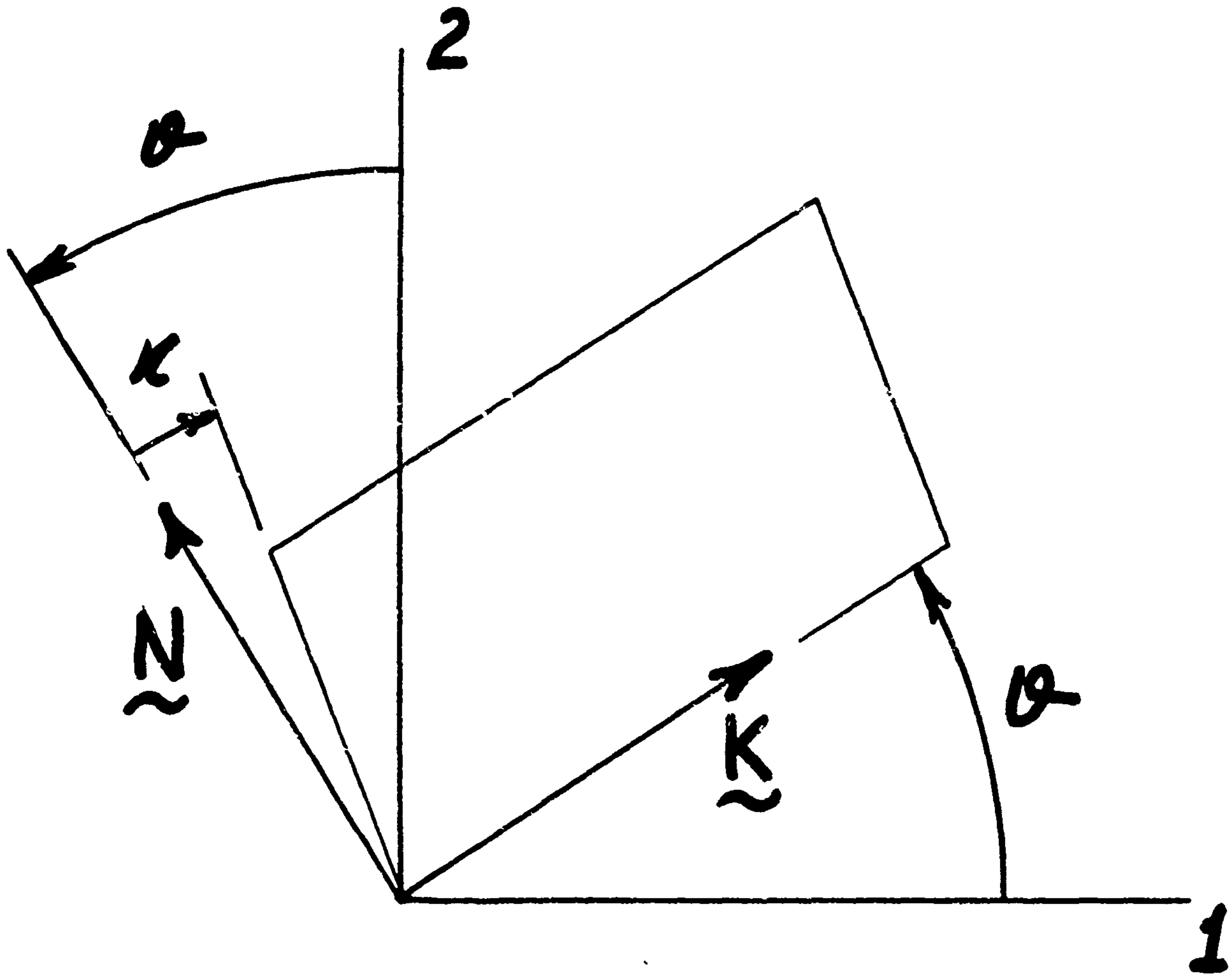

Figure 1. Sawyers and Rivlin